Is targeted advertising always beneficial?

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Abstract

In this paper, we study a simple model in which two horizontally differentiated firms compete in prices and targeted advertising on an initially uninformed market. First, the Nash equilibrium is fully characterized. We prove that when the advertising cost is low, firms target only their “natural market”, while they cross-advertise when this cost is high. Second, the outcome at equilibrium is compared with random advertising. Surprisingly, we prove that firms’ equilibrium profits may be lower with targeted advertising relative to random advertising, while firms are given more options with targeted advertising.

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1 Introduction

When a firm launches a new product, it uses informative advertising to generate demand for the product and to make potential customers aware of the existence of the new product, its attributes and its price. For a long period firms had been unable to discriminate their advertising expenditures between the different groups of consumers, either because of lack of information on consumers’ habits or because they had no means to reach some of them without reaching the others. Hence they had been using random (mass) advertising such as newspapers advertisements or general television channels which allow them to reach randomly different types of consumers. However in doing so, the firm may spend its money in sending messages to a lot of people who will probably never buy the product even when they are well informed about it.

Nowadays, on the one hand, there is a large proliferation of specialized media oriented to specific segments of the market such as magazines, private TV channels focused on sport, health, food, cars... On the other hand, the outstanding growth of internet social networking such as Facebook, MySpace and LinkedIn, enable firms to obtain a better information on consumers’ preferences. It has thus become possible for firms to focus on the most interesting consumers and avoid wasteful mass advertising.

The aim of this paper is double. First, to what extent will a firm choose to target its “natural consumers”? Second, is targeted advertising always beneficial to firms relative to random advertising?

It has been generally accepted as obvious that targeted advertising increases equilibrium profits. This view is very clearly expressed for instance in Iyer et al. (2005) who conclude that “... the ability to target advertising provides benefits that are not lost when competitors respond by implementing targeting of their own. Because of reduced waste, targeted advertising can simultaneously make all firms better off”. The present paper shows that this outcome is not warranted. In our setting, the transition from mass to targeted advertising may result in a reduction of profits for both firms.

We develop a horizontal differentiation model à la Hotelling where consumers are uniformly distributed and the firms are located at the two extremities of a “linear city”. As in Grossman and Shapiro (1984) or the simplified version of Tirole (1988), potential customers are not initially aware of the existence of the firms. Each firm chooses its price and its advertising strategy in order to inform consumers about its product. Our approach differs however from the previous ones as we assume that firms are able to perfectly target their consumers, choosing a different advertising strategy for each type of consumers. We investigate the Subgame Perfect Nash Equilibrium in this case and compare the equilibrium outcome to the one obtained in the random advertising model of Tirole (1988).

Main results. First we show that even though each firm is able to target each type of consumers, it chooses at equilibrium to differentiate its advertising strategies only between its natural market, (in which consumers, if fully informed, would buy from this firm) and the natural market of the rival’s one. Our second main finding concerns the market equilibrium. We show that for sufficiently low advertising cost, the market is perfectly segmented at equilibrium with each firm targeting only its natural consumers. In this case, the equilibrium prices are larger and firms achieve higher profits than those under random advertising.
When the advertising cost is high enough, firms cross-advertise. Each firm differentiates its targeting strategy in order to reach a fraction of its natural consumers and a lower but positive fraction of the natural consumers of its rival. In the latter case, the equilibrium prices are lower under targeted advertising than under random advertising.

**Related literature.** Informative advertising has been thoroughly studied. A major result in this context is due to Butters (1977) who studies price advertising in markets for homogeneous goods. An important contribution by Grossman and Shapiro (1984) extended the Butters’ model and introduced product differentiation via a circle model to show how informative advertising affects price competition in an oligopoly market when products are horizontally differentiated and advertising is uniform throughout the market. Several papers adapted the Grossman and Shapiro’s (1984) model to analyze different market configurations where consumers can buy only products on which they are informed, such as Celik (2007), Hamilton (2004), Bester and Petrakis (1995), etc.

Several authors have later developed models in order to formally establish the effect of targeted advertising on prices and competition. Most of this research has assumed that firms directly target different groups of consumers. Our framework is very close to the one developed by Tirole (1988) who adapted Grossman and Shapiro’s model to fit a linear city. However, none of these works on targeted advertising have assumed that a firm is allowed to target and reach a fraction of the rival’s consumers, an assumption which makes the originality of our model.

Two different arguments have been put forward to explain the superiority of targeted advertising upon mass or random advertising:

1. The first and simplest one refers to cost savings. According to these studies, targeting allows to save on advertising expenditures towards consumers who will never be willing to buy the firm’s product and to advertise more towards the others. For instance, Esteban et al. (2001) and Hernandez-Garcia (1997) argue, in a monopolist framework, that the overall level of advertising falls with targeting and show similarly to Iyer et al. (2005), that the use of targeted advertising increases the market price and leads to higher profits in comparison to mass or random advertising.

2. The second argument is that targeting may be an implicit collusion device between firms producing differentiated goods since each firm will advertise at equilibrium only towards its consumers. In particular, Galleotti and Gonzalez (2004) with a homogeneous product competition as well as Iyer et al. (2005) with horizontally differentiated products, find that there is only a Nash equilibrium in mixed strategies therefore, that targeting can fragment the market only from time to time. Furthermore, Roy (2000) shows through a sequential targeted pricing-advertising game, that at equilibrium, the entire market is divided into mutually exclusive captive segments where each firm acts as a pure local monopolist when the cost of advertising approaches zero. In the same way, in a vertical differentiation product framework, Esteban and Hernandez (2007) derive a Nash equilibrium in pure strategies and claim similarly to the previous authors that targeted advertising can lead to a market fragmentation into local monopolies.
Our model shares with the second set of literature the result on perfect segmentation of the market only for low advertising cost. When the costs are high, the market is not perfectly segmented. Moreover, the increase of prices and firms’ profits relative to random advertising, obtained by Esteban et al. (2001), Hernandez et al. (1997) as well as Iyer et al. (2005), holds in our model only for low advertising cost.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 provides a detailed analysis of the equilibrium of the market. In section 4 the outcome of the model is compared with random advertising. Finally, section 5 concludes the paper. Proofs and intermediate results are relegated to an appendix each time it improves the readability of the paper.

2 The model and some preliminary results

We consider a Hotelling model, with two firms competing in prices and advertising. Firms are exogenously located at opposite endpoints of a linear city of length 1 (without any loss of generality, Firm 1 is located at 0 and Firm 2 at 1 on the unit interval $[0,1]$). They produce homogeneous goods at the same cost $c$ and offer them at price $p_i$ ($i = 1, 2$). Firms invest in advertising to inform consumers about their existence, products’ characteristics and prices.

A unit mass of consumers are uniformly distributed along the linear city. They have heterogeneous preferences for products’ attributes ($x$) and are initially unaware of the existence of either firm (they do not search for information about products). Consumers have no source of information other than advertisements. They seek to purchase the products that best fit their needs\(^1\) among the products on which they have information. Each consumer derives a gross utility $U_0 > 0$ from consuming at most one unit of an ideal product\(^2\). On visiting one of the two firms, each consumer incurs a transportation cost linear w.r.t the distance between the consumer and the seller’s location, with rate $t$. A consumer of type $x \in [0,1]$ has the following utility functions when buying from Firm 1 or Firm 2 respectively:

$$\begin{align*}
U_1(x) &= U_0 - tx - p_1 \\
U_2(x) &= U_0 - t(1 - x) - p_2
\end{align*}$$

We define $\hat{x}$ as the marginal consumer who, when totally informed of the existence of both products, is indifferent between purchasing from Firm 1 and from Firm 2:

$$\hat{x} = \frac{(t - p_1 + p_2)}{2t}. \quad (1)$$

In the following we shall say that the segment $[0, \hat{x}]$ is the “natural market” of Firm 1 while the segment $[\hat{x}, 1]$ is the “natural market” of Firm 2. Obviously firms can influence the “natural market” sharing by changing their prices. As we shall see, this leads them to change accordingly their advertising strategies.

\(^1\)This assumption follows Butters (1977) and Grossman and Shapiro (1984).

\(^2\)We assume that the consumer’s valuation $U_0$ is sufficiently large to ensure that a consumer purchases if any ad is received.
Let \( g_i(x) \) be the advertising intensity function, i.e. the proportion of type \( x \)-consumers informed of the existence of Firm \( i \). The cost of reaching fraction \( g_i(x) \) of consumers is assumed to be:

\[
A(g_i(x)) = \frac{a}{2} (g_i(x))^2, \tag{2}
\]

with a maximum advertising expenditure of \( \frac{a}{2} \).

When a firm chooses its advertising expenditure, it equivalently chooses its served demand. If Firms 1 and 2 advertise at levels so that fractions \( g_1(x) \) and \( g_2(x) \) of consumers are reached respectively, then a fraction \( (1 - g_1(x))(1 - g_2(x)) \) of consumers receives no ad and stays out of the market. A fraction \( g_1(x)(1 - g_2(x)) \) receives only Firm 1’s ads and buys from Firm 1. Likewise, a fraction \( g_2(x)(1 - g_1(x)) \) buys from Firm 2. Finally, a fraction \( g_1(x)g_2(x) \) of consumers who receive ads from both firms, are fully informed and buy their most preferred product.

We analyze a two-step game in which the two firms first choose simultaneously their prices \( p_i \) in \( [0, 3t + c] \) then their advertising levels

\[ g_i(x) \text{ with } i = 1, 2, \]

We hence suppose that firms can perfectly target their advertising expenditures, i.e. Firm \( i \) is able to choose a different value of \( g_i(x) \) for each \( x \in [0, 1] \).

The following Lemma provides the possible shape of advertising strategies at equilibrium.

**Lemma 1** At equilibrium, the advertising intensity \( g_i(x) \), with \( i = 1, 2 \), is a step-function which takes two constant values respectively on \([0, \hat{x}]\) and \([\hat{x}, 1]\).

Lemma 1 states that firms do not have incentives to differentiate their advertising intensity for each consumer-type in the linear city. At equilibrium, each firm chooses to differentiate its targeted advertising only between its “natural market” and the “natural market” of its rival.

From Lemma 1, the choice of advertising strategies amounts to the choice of \((\Phi_i, \Omega_i)\) where \( \Phi_i \) corresponds to the fraction of consumers targeted in its “natural market” \( \Phi_1 \) for Firm 1 on \([0, \hat{x}]\) and \( \Phi_2 \) for Firm 2 on \([\hat{x}, 1]\) and \( \Omega_i \) in its rival’s one \( \Omega_1 \) for Firm 1 on \([\hat{x}, 1]\) and \( \Omega_2 \) for Firm 2 on \([0, \hat{x}]\).

### 3 Targeted advertising

As a result of the definitions and Lemma 1, the profit of Firm \( i \) writes as:

\[
\pi_i = \begin{cases} 
  m_i \Phi_i - \frac{a}{2} \Phi_i^2 & \text{if } m_i < m_j - t \\
  m_i (\Phi_i \hat{x} + \Omega_i (1 - \Phi_j) (1 - \hat{x})) - \frac{a}{2} \Phi_i^2 \hat{x} - \frac{a}{2} \Omega_i^2 (1 - \hat{x}) & \text{if } m_j - t < m_i < m_j + t \\
  m_i \Omega_i (1 - \Phi_j) - \frac{a}{2} \Omega_i^2 & \text{if } m_i > m_j + t 
\end{cases}
\]

where

\[
\hat{x} = \frac{(t - m_i + m_j)}{2t}.
\]

---

\(^3\)The sequentiality of the game is chosen for expositional clarity. We prove in the appendix (Lemma 10) that the sequential equilibrium coincides with the equilibrium of the game where the firms choose simultaneously their prices and advertising strategies.
When $m_i < m_j - t$, Firm $i$’s natural market corresponds to the whole segment $[0, 1]$. But the served demand depends on its advertising strategy $\Phi_i$. When $m_j - t < m_i < m_j + t$, the marginal consumer indifferent between the two firms is between 0 and 1, so that each firm has a positive natural market share. Finally, when $m_i > m_j + t$, Firm $i$ has a null natural market share but it may make profit from the ill-informed consumers.

Lemma 2 solves the second step of the game corresponding to the choice of the advertising strategies, for given markups, in each case of Equation (3).

**Lemma 2** For given $m_i$ and $m_j$, define $\Phi_i^*(m_i, m_j)$ and $\Omega_i^*(m_i, m_j)$ to be the advertising strategies at the Nash equilibrium of the second step.

- For $m_i < m_j - t$, we have $\Phi_i^*(m_i, m_j) = \min(\frac{m_i}{a}, 1)$ and $\Omega_i^*(m_i, m_j)$ may take any value in $[0, 1]$. 
- For $m_j - t < m_i < m_j + t$,
  \[
  \begin{cases} 
  \Phi_i^*(m_i, m_j) = \min(\frac{m_i}{a}, 1) \\
  \Omega_i^*(m_i, m_j) = \max(0, \min(1, \frac{m_i(a-m_j)}{a^2}))
  \end{cases}
  \]
- For $m_i > m_j + t$, $\Phi_i^*(m_i, m_j)$ may take any value in $[0, 1]$, $\Omega_i^*(m_i, m_j) = \max(0, \min(1, \frac{m_i(a-m_j)}{a^2}))$ and $\Phi_i^*(m_i, m_j)$ may take any value in $[0, 1]$.

Note that we always have $\Phi_i^* \geq \Omega_i^*$ thus each firm spends more in advertising on its own natural market. In other words, on the one hand Firm $i$ is incited to make an important investment in advertising in order to inform its own natural consumers. Even though these consumers might also be informed of the existence of Firm $j$, they will choose to buy the product of Firm $i$. On the other hand, Firm $i$ chooses also to reach a small fraction of the competitor’s potential consumers, because it tries to capture some consumers from its rival that might not be informed of the existence of Firm $j$’s product. However, Firm $i$ will never choose to make a higher level of advertising for the rival’s natural consumers, since it will result in a too important waste of advertising expenditures relative to the gain in market share.

For the remaining of the analysis, we define an interior equilibrium as an equilibrium such that: $\Phi_i^*, \Omega_i^* \in [0, 1]$. Accordingly, an interior symmetric equilibrium such that $m_i = m_j = m$, must satisfy the F.O.C w.r.t prices evaluated at the above values of $\Phi_i$ and $\Omega_i$. This leads to the following third-order polynomial:

\[ P(m) = m^3 - 2m^2(a-t) - 4amt + 4a^2t = 0. \] (4)

Furthermore, we define a corner equilibrium as an equilibrium such that: $\Phi_i = 1$ for at least one Firm $i$. Lemma 3 is a technical one needed to identify the equilibrium in prices.

**Lemma 3** Two cases have to be distinguished for the third degree polynomial $P(m)$:
When $a > 2t$, $P(m)$ admits one negative root and two positive ones. Denote by $m^*$ the smallest positive root. We have:

1. $2t < m^* < t + \frac{a}{2} < a$.
2. $P(m^*) = 0; \forall m \in [0, m^*), \ P(m) > 0; \forall m \in (m^*, a], \ P(m) < 0$.

When $a \leq 2t$, $\forall m \in [0, a], \ P(m) > 0$.

We now solve the first step of the game corresponding to the choices of markups thus characterize fully the SPNE. It turns out that the equilibrium is different depending on the value of $a$, the advertising cost parameter.

**Proposition 1** The SPNE of the game depends on the level of advertising cost $a$.

- If $a > 2t$, the unique equilibrium is an interior symmetric equilibrium such that each firm $i$ targets a fraction $\Phi_i$ of its natural market and a lower fraction $\Omega_i$ of the natural market of its competitor. This equilibrium is given by:

\[
\begin{align*}
\Phi_1 &= \Phi_2 = \Phi^* = \frac{m^*}{a}, \\
\Omega_1 &= \Omega_2 = \Omega^* = \frac{m^*(a-m^*)}{a^2}
\end{align*}
\]

where $m^* = m(a, t)$ defined in Lemma 3. The corresponding profit at equilibrium is given by:

\[
\pi^* = \frac{m^{*2}(a^2 + (a - m^*)^2)}{4a^3}.
\]

- If $a \leq 2t$, the unique equilibrium is a corner symmetric equilibrium such that the market is perfectly segmented, i.e each firm informs all its natural consumers and does not target the natural consumers of its rival. The equilibrium advertising and pricing strategies are given by:

\[
\begin{align*}
\Phi_1 &= \Phi_2 = \Phi^* = 1, \\
\Omega_1 &= \Omega_2 = \Omega^* = 0, \\
m_1 &= m_2 = m^* = t + \frac{a}{2}
\end{align*}
\]
The corresponding equilibrium profit, given by:

\[ \pi^* = \frac{t}{2}, \quad (8) \]

is independent of the advertising cost.

When the advertising cost is low \((a \leq 2t)\), each firm invests in advertising to inform its whole natural consumers while ignoring the natural consumers of the competitor. Obviously, the low advertising cost pushes firms to invest more in advertising. But expecting the same behavior from the competitor, the firm has no interest to target the competitor’s natural market which is not possible to capture when fully informed by the competitor. Thus targeting the latter segment would result in a pure waste of advertising expenditure thus would not be profitable. In this case, the cost of advertising is fully reflected in the price of the product and passed on to customers. This is why the profit is independent of the advertising cost \(a\).

When advertising cost is high enough, each firm decreases the advertising intensity toward its natural consumers (\(\Phi\)) and targets a fraction of its rival’s natural consumers (\(\Omega\)). If the advertising cost increases, by a direct effect firms reduce their advertising investment. Each firm expecting its competitor’s reduction of advertising on its own market, will be interested by attracting some ill-informed consumers from that market. However, if the advertising cost increases significantly and reaches high levels, the advertising effort of each firm to capture a share of the natural consumers of its competitor decreases, as the loss of profit resulting from the high advertising expenditures outweighs the gain obtained from the additional acquired market share.

![Figure 3: Advertising Equilibrium levels](image1)

![Figure 4: Equilibrium Profit](image2)

**Lemma 4** When \(a > 2t\), firms’ equilibrium prices are increasing with the cost of advertising \(a\):

\[ \frac{dm^*}{da} > 0. \quad (9) \]

The increase of advertising cost reduces the overall level of advertising of each firm. Hence the informational product differentiation increases, allowing firms to relax competition and raise their prices.
From Fig. 4, surprisingly the profit is decreasing for sufficiently low levels of $a$ then becomes increasing. The shape of the profit may however be explained as follows. Raising $a$ has a negative direct effect by raising the advertising cost and a positive indirect one as it allows higher prices. The indirect effect outweighs the direct one for high levels of $a$.

Our results are only partially in accordance with the findings of the previous studies made on targeted advertising. Indeed the result of perfect segmentation of the market for low levels of advertising cost is consistent with Roy (2000) who investigates a sequentially advertising-pricing game for homogeneous products and shows that each firm acts as a pure local monopolist in its captive segment when the cost of advertising approaches zero. However in our model, for high levels of advertising cost, the market is not divided into exclusive captive segments, while for the most part of these works, targeting increases the extent of market segmentation and raises the monopoly power, such as Galleotti et al. (2004) with homogeneous products, Iyer and al. (2005) with horizontally differentiated products and Esteban et al. (2007) with vertically differentiated products.

4 Comparison with random advertising

In this section, we compare the outcome at equilibrium with the outcome with random advertising, in terms of prices, advertising costs and profits. To do so we first recall the main results of Tirole’s model.

Tirole (1988) assumes that Firm $i$ ($i = 1, 2$) has only the possibility to invest in random advertising, i.e. to reach uniformly all consumers through a constant function $g_i$:

$$g_i(x) = \Psi_i \text{ for all } x \in [0, 1]. \quad (10)$$

He established the following proposition.

**Proposition 2 [Tirole 1988]** The SPNE of the random advertising game depends on the level of advertising cost $a$.

- If $a > \frac{t}{2}$, there is a unique symmetric equilibrium given by:

$$\begin{cases} 
p_1^r = p_2^r = p^r = c + \sqrt{2at} \\
\Psi_1^r = \Psi_2^r = \Psi^r = \frac{2}{1 + \sqrt{\frac{2a}{t}}} 
\end{cases}, \quad (11)$$

with the corresponding equilibrium profit:

$$\pi^r = \frac{2a}{\left(1 + \sqrt{\frac{2a}{t}}\right)^2}. \quad (12)$$

- If $a \leq \frac{t}{2}$, there is a unique full information symmetric equilibrium where

$$\begin{cases} 
p_1^r = p_2^r = p^r = c + t \\
\Psi_1^r = \Psi_2^r = \Psi^r = 1 
\end{cases}, \quad (13)$$
and the corresponding equilibrium profit:

\[ \pi^r = \frac{t - a}{2}. \]  

(14)

We now compare the equilibrium outcomes with targeted advertising and with random advertising. The comparison is done with several respects: advertising strategies, prices, costs and firms’ profits.

Advertising strategies. Under targeted advertising, firms have the possibility to advertise differently to both types of segments. Each firm focuses more on its natural market and reduces its efforts on the rival’s market, relative to random advertising where firms are constrained to advertise uniformly on both segments. It is thus natural to have \( \Omega^* \leq \Psi^* \leq \Phi^* \) (Fig. 5).

In addition, note that the fraction of natural consumers of a firm who are informed only by the rival firm is more important under random advertising relative to targeted advertising (Fig. 6). Thus target advertising is more beneficial for consumers since it allows an efficient information product coverage as firms target more their natural market than that of their rival which implies that a larger proportion of consumers will buy their preferred product under target advertising relative to the random case.

Figure 5: Equilibrium advertising strategies

Figure 6: Rival’s captive market

Prices. Note that prices are for the same reason in both cases increasing with \( a \). The comparison between equilibrium prices in the two regimes leads to the following Lemma.

Lemma 5 The equilibrium prices under targeted advertising are larger than under random advertising when \( a \leq 2t \) and lower when \( a \geq 2t \).

Proof. (i) When \( a \leq \frac{t}{2} \), \( p^r = c + t < p^* = c + t + \frac{a}{2} \);

(ii) when \( a \in \left[ \frac{t}{2}, 2t \right] \), \( p^r = c + \sqrt{2at} \leq p^* = c + t + \frac{a}{2} \) since \( p^* - p^r = (\sqrt{t} - \sqrt{\frac{a}{2}})^2 \);

(iii) when \( a \geq 2t \), \( p^r \geq p^* \) since \( P(\sqrt{2at}) = 2at (2t - \sqrt{2at}) \leq 0 \).

The intuition behind this result is quite simple. First, when the advertising cost is low enough, as we already showed, firms focus only on their natural consumers under targeted advertising strategy. They do not make cross-advertising, i.e. each consumer receives an ad only from its preferred product. This implies a perfect informational
product differentiation, allowing firms to raise their prices significantly and to act as local monopolists. This advertising behavior is more efficient than the firms’ behavior under random advertising because in the latter case, firms reach consumers that might not be interested by their product and thus, achieve less informational product differentiation and are less able to raise prices.

Second, when the advertising cost is sufficiently high, a price cut allows the undercutting firm (say Firm 1) to attract more additional buyers as the number of its natural market increases by $\Delta \hat{x}$. Notice that the natural market increases in the same proportion whatever is the advertising strategy. The fraction of $\Delta \hat{x}$ who now buy at Firm 1 is equal to $\Phi_1$ under targeted advertising and $\Psi_1$ under random advertising. The fraction of $\Delta \hat{x}$ who previously bought at Firm 1 is equal to $\Omega_1(1 - \Phi_2)$ under targeted advertising and $\Psi_1(1 - \Psi_2)$ under random one. It follows then that the same price cut attracts $\Delta \hat{x}(\Phi_1 - \Omega_1(1 - \Phi_2))$ additional buyers under targeted advertising and $\Delta \hat{x}(\Psi_1 - \Psi_1(1 - \Psi_2))$ under random advertising. We easily check that $\Delta \hat{x}(\Phi_1 - \Omega_1(1 - \Phi_2)) > \Delta \hat{x}(\Psi_1 - \Psi_1(1 - \Psi_2))$ since $\Phi_1 > \Psi_1$ (Fig. 5) and $\Omega_1(1 - \Phi_2) < \Psi_1(1 - \Psi_2)$ (Fig. 6). Thus, for high advertising cost, price competition is fiercer under targeted advertising than under random advertising. This results in lower equilibrium prices in the former case.

![Figure 7: Random vs. Targeted prices](image)

**Costs.** Fig. 8 shows that with random advertising, the advertising expenditures are increasing with $a$. However with targeted advertising, advertising expenditures begin increasing with $a$ until reaching $a = 2t$ then decrease in a small range of $a$ close to $2t$ to finally increase w.r.t the advertising cost. The radical change of structure under targeted advertising when $a$ reaches $2t$ is due to the fact that firms move from a situation of local monopolists to a situation of cross-advertising. This configuration explains the curve of advertising costs in the neighborhood of $a = 2t$.

Visibly, under targeted advertising the total advertising cost of each firm is “almost all the time” lower than the one under random advertising (Fig. 8). This intuitive result comes in accordance with the findings of Esteban et al. (2001), Hernandez-Garcia (1997) and Iyer et al. (2005). Our result emphasizes the motivation of firms to targeted advertising since it allows them to make cost savings and efficient advertising.

Recall that $\hat{x}$ (defined by Expression (1)) is the marginal consumer who, when totally informed of the existence of both products, is indifferent between purchasing from Firm 1 and from Firm 2.
by avoiding the consumers not interested in their products and advertising towards the most interesting consumers.

Note however that the advertising expenditures are larger with targeted advertising relative to random advertising in a small segment around $a = 2t$. Indeed for $a \in [0, 2t]$, under targeted advertising, each firm informs its natural market with a maximal advertising intensity ($\Phi^* = 1$). Under random advertising, for $a \in [0, \frac{t}{2}]$, each firm informs the whole market with a maximal advertising intensity ($\Psi^r = 1$). Thus, on the range $[0, \frac{t}{2}]$ advertising expenditures under random advertising are clearly larger than those under targeted advertising. For $a > \frac{t}{2}$ each firm continues under random advertising to reach the whole market but with a lower advertising intensity ($\Psi^r < 1$) while each firm reaches only its natural market with maximal intensity under targeted advertising. As the random advertising intensity is decreasing w.r.t the advertising cost, there is some threshold after which the advertising expenditures under random advertising become lower than those achieved under targeted advertising. The loss from targeted advertising reaches a maximum for $a = 2t$. To benefit again from targeted advertising, $a$ must go beyond another threshold.

![Figure 8: Random vs. Targeted advertising costs](image)

**Profits.**

Note that profits are in both cases decreasing then increasing. In both cases the positive strategic effect outweighs the direct one for sufficiently high advertising costs. The most striking result, is that for large values of the advertising cost, targeted advertising may reduce firms’ equilibrium profits relative to their random advertising equilibrium level. Indeed, as far as the advertising cost is low relative to the parameter of horizontal differentiation, targeting increases firms’ equilibrium profits since it allows them to save on advertising costs by concentrating their advertising effort on their natural customers. But for large values of the advertising cost, targeting leads to lower equilibrium prices and this negative strategic effect on firms’ profits dominates the positive direct one.

Consequently, for high levels of advertising cost, firms would mutually benefit if they could jointly decide to use random advertising as they would achieve higher prices and profits. The freedom allowed by targeted advertising is not beneficial for firms as it is Pareto dominated by the equilibrium under random advertising. But random
advertising by each firm is not an equilibrium under targeted advertising. Therefore firms may wish to have less options, i.e. they may wish that a third party restrict advertising to random one, forbidding targeted advertising.

Figure 9: Random vs. Targeted firms' profits

Our result partially contradicts Iyer et al. (2005) who argue that targeting always increases firms’ prices and profits with comparison to uniform advertising. Indeed, Iyer et al. (2005) assume that targeting is only oriented to the consumers who have strong preferences for their product. Similarly, Esteban et al. (2001) argue that a monopolist will direct heavier advertising weights to the consumers who are willing to pay more for the product, and that the overall level of advertising falls with targeting and the market price increases. Our result is in accordance with their finding if and only if the advertising cost is low. In this case, the advertising reaches only the most interesting consumers, leading to less wasteful advertising and allowing profits higher than in the case of random advertising. When the advertising cost is high, targeting reduces firms’ profits relative to random advertising. This outcome is possible in our paper and not in the previous ones, because unlike them, we suppose that each firm has the possibility to target the natural consumers of the competitor.

5 Conclusion

In this paper we have presented a modified duopoly version of Grossman and Shapiro (1984), the seminal paper on informative advertising. We analyzed the transition from random advertising to targeted advertising and investigated through this model the benefits of targeted advertising for firms.

This paper has argued that, when firms have the ability to target each type of consumers, each firm chooses at equilibrium to differentiate its advertising strategies between its natural market and its rival’s one. The full characterization of price-advertising equilibrium gives rise to two possible cases depending on the advertising cost relative to the transportation cost. When the advertising cost is low, each firm targets only its natural market, ignoring the other. However when the advertising cost is high, each firm targets both markets but in different proportions. Surprisingly, we
find that random advertising may pareto-dominate targeting for firms for high values of advertising, which comes in contrast to the previous studies on targeted advertising.

Several perspectives are possible to develop the present paper. First, welfare implications may be investigated, comparing the outcome obtained under targeting with the optimal outcome. Second, by introducing media competition, the advertising cost becomes endogenous, which would bring new insights on the relation between product and media competition. Finally we may study the incentives of firms for concentration considering the competitive framework of Grossman and Shapiro (1984) with \( n \) firms located on a circular city having the ability to target each type of consumers.

6 Appendix

Proof of Lemma 1.

Firm 1’s profit function is given by:

\[
\pi_1 = (p_1 - c) \left[ \int_0^{\hat{x}} g_1(x) \, dx + \int_{\hat{x}}^1 g_1(x)(1 - g_2(x)) \, dx \right] - \frac{a}{2} \int_{\hat{x}}^1 (g_1(x))^2 \, dx. \tag{15}
\]

We choose \( g_1(x) \) that maximizes Firm 1’s profit respectively when \( x \in [0, \hat{x}] \) and \( x \in [\hat{x}, 1] \), which implies:

\[
\begin{align*}
\frac{\partial \pi_1}{\partial g_1(x)} &= p_1 - c - ag_1(x) \quad \text{for all } x \in [0, \hat{x}] \\
\frac{\partial \pi_1}{\partial g_2(x)} &= (p_1 - c)(1 - g_2(x)) - ag_1(x) \quad \text{for all } x \in [\hat{x}, 1]
\end{align*}
\tag{16}
\]

and symmetrically for Firm 2:

\[
\begin{align*}
\frac{\partial \pi_2}{\partial g_1(x)} &= (p_2 - c)(1 - g_1(x)) - ag_2(x) \quad \text{for all } x \in [0, \hat{x}] \\
\frac{\partial \pi_2}{\partial g_2(x)} &= p_2 - c - ag_2(x) \quad \text{for all } x \in [\hat{x}, 1]
\end{align*}
\tag{17}
\]

From these conditions, we deduce that \( g_1(x) \) is constant and equal to \( \Phi_1 = \min \left\{ \frac{p_1 - c}{a}, 1 \right\} \) on \([0, \hat{x}]\) and \( g_2(x) \) is constant and equal to \( \Phi_2 = \min \left\{ \frac{p_2 - c}{a}, 1 \right\} \) on \([\hat{x}, 1]\). Consequently, \( g_1(x) \) is constant and equal to \( \Omega_1 = \frac{(p_1 - c)(1 - \Phi_2)}{a} \) on \([0, \hat{x}]\) and \( g_2(x) \) is constant and equal to \( \Omega_2 = \frac{(p_2 - c)(1 - \Phi_1)}{a} \) on \([\hat{x}, 1]\). \( \blacksquare \)

Proof of Lemma 2.

• When \( m_i < m_j - t \): From expression 3, \( \pi_i = m_i \Phi_i - \frac{a}{2} \Phi^2 \). For a fixed \( m_i \), first order condition w.r.t. \( \Phi_i \) yields \( \Phi_i = \frac{m_i}{a} \), thus the result.

• For the second case of Equation (3): \( m_j - t < m_i < m_j + t \), the profit is given by Equation (3) as a function of \( (\Phi_i, \Omega_i) \). The gradient of the profit function w.r.t. these variables for a given \( m_i \) writes as:

\[
\begin{align*}
\frac{\partial \pi_i}{\partial \Phi_i} &= -\frac{1}{2t} \left( t - m_i + m_j \right) (-m_i + a\Phi_i) \\
\frac{\partial \pi_i}{\partial \Omega_i} &= -\frac{1}{2t} \left( t + m_i - m_j \right) ((\Phi_j - 1)m_i + a\Omega_i)
\end{align*}
\]
First order conditions are satisfied for:

\[
\begin{align*}
\Phi_i &= \Phi_i = \frac{m_i}{a(a-m_j)} \\
\Omega_i &= \Omega_i = \frac{m_i(a-m_j)}{a^2}
\end{align*}
\]

The hessian matrix relative to these two variables writes:

\[
\begin{pmatrix}
\frac{-a(m_j-m_i+t)}{2t} & 0 \\
0 & \frac{-a(m_i-m_j+t)}{2t}
\end{pmatrix}
\]

which is definite negative. Thus if \((\Phi_i, \Omega_i) \in [0,1] \times [0,1]\), it realizes the global maximum of the profit. Otherwise the global maximum corresponds to a corner solution.

- When \(\Phi_i > 1\), we have \(\frac{\partial \pi}{\partial \Phi_i} > 0\), for all \(\Phi_i \in [0,1]\). Thus \(\Phi_i = 1\) realizes the maximum of \(\pi_i\). Hence \(\Phi_i^* = \min\left(\frac{m_i}{a}, 1\right)\) realizes the maximum of \(\pi_i\).
- Take now the optimal value \(\Phi_j^*\). For \(\Phi_j^* = 1\) (i.e. \(m_j > a\)), \(\frac{\partial \pi}{\partial \Phi_j} < 0\) thus necessarily \(\Omega_j = 0\). For \(\Phi_j^* = \frac{m_j}{a} < 1\) (i.e. \(m_j < a\)), \(\frac{\partial \pi}{\partial \Phi_j} < 0\) when \(\Omega_j > \Omega_i\) and positive otherwise thus reaches its maximum at \(\Omega_i^*\).

- For the third case of Equation (3): \(m_i > m_j + t\), \(\pi_i = m_i \Omega_i (1 - \Phi_j) - \frac{a}{2} \Omega_i^2\). For a fixed \(m_i\), F.O.C w.r.t \(\Omega_i\) yields \(\Omega_i = \frac{m_i(a-m_j)}{a^2}\). The profit is independent of \(\Phi_i\).

**Proof of Lemma 3.** We have \(P'(m) = 3m^2 - 4m(a-t) - 4at\). \(P'(m) = 0\) for \(m = m_1 < 0\) or \(m = m_2 > 0\). Considered for \(m \geq 0\), \(P\) is decreasing up to \(m = m_2\) then becomes increasing.

- When \(a > 2t\), we have \(2t < t + \frac{a}{2} < a\). On the one hand, \(P(2t) = 4t(a-2t)^2 > 0\), \(P(t + \frac{a}{2}) = \left(\frac{2t-a}{8}\left(a+4(3a-4)+12t\right)\right) < 0\) and \(\lim_{m \to +\infty} P(m) = +\infty\). Hence \(P\) admits two positive roots. One of them \(2t < m^* < t + \frac{a}{2}\). On the other hand \(P(a) = -a^2(a-2t) < 0\) thus the largest root is necessarily greater than \(a\) (Fig. 1).

- When \(a \leq 2t\). We check that in this case, \(P(a) = a^2(2t-a) > 0\) and \(P'(a) = -a^2 < 0\). The two positive roots of \(P(m)\), when they exist, are larger than \(a\). As a result, \(P(m) > 0\) for all \(m \in [0,a]\) (see Fig. 2).

**Proof of Proposition 1.**

The analysis has to be made for the two cases of high and low advertising cost: \(a > 2t\) and \(a \leq 2t\). In each case, the proof is made through a series of Lemmas.

**Case 1: \(a > 2t\)**

**Lemma 6** Assume that \(a > 2t\). Let \(\phi_i(m_j)\) be the best reply of Firm \(i\) to \(m_j\) and denote by: \(\Psi(m_j) = m_j + \frac{P(m_j)}{3m_j(2a-m_j)}\). We have the following results.

\(i\) \(\forall m_j \in [0,m^*], \phi_i(m_j) > m_j\).

\(ii\) \(\forall m_j > m^*, \phi_i(m_j) < m_j\).

\(iii\) \(\phi_i(m^*) = \Psi(m^*) = m^*\).

\(iv\) \(\forall m_j \in [m^*, \infty), \phi_i'(m_j) > 0\).
Proof. We replace in Firm $i$’s profit given by Equation (3) the equilibrium advertising strategies by $(\Phi_i^*, \Omega_i^*, \Phi_j^*, \Omega_j^*)$ defined in Lemmas 2 and 3.

Note that in all cases when $m_i \leq m_j - t$, we have $\Phi_i^* = \frac{m_i}{a}$ as $m_i < m^* < a$. The profit of Firm $i$, $\pi_i = \frac{m_i^2}{2a}$ is strictly increasing w.r.t $m_i$.

(i) Suppose $m_j \in [0, m^*)$:

When $m_i \in [m_j - t, \min\{a, m_j + t\}]$, we have $\Phi_i^* = \frac{m_i}{a}$, $\Phi_j^* = \frac{m_j}{a}$ and $\Omega_i^* = \frac{m_i(a-m_j)}{a^2}$. Thus $\pi_i = \Gamma(m_j) = \frac{m_i^2 + m_j^2}{2m_{i,j}} - \frac{m_i^2}{2m_{i,j}} - \frac{m_j^2}{2m_{i,j}} - \frac{3m_i^2}{2m_{i,j}} + \frac{3m_j^2}{2m_{i,j}} - \frac{3m_i^2}{2m_{i,j}} + \frac{3m_j^2}{2m_{i,j}}$. Considered for unconstrained $m_i$, $\Gamma(m_i)$ reaches a global maximum at $\Psi(m_j)$. Note that $\Psi(m_j) > m_j$ for $m_j < m^*$ (Lemma 3) and that $\min\{a, m_j + t\} > m_j$.

$\pi_i$ reaches its maximum on $[0, \min\{a, m_j + t\}]$ either at $\Psi(m_j)$ (if $\Psi(m_j) < \min\{a, m_j + t\}$) or at $\min\{a, m_j + t\}$. In both cases, the maximum on $[0, \min\{a, m_j + t\}]$ is greater than $m_j$, thus $\varphi_i(m_j) > m_j$.

(ii) When $m_j > m^*$ we have $m_j + t > m^* + t > 3t$ (Lemma 3). Thus, for all $m_i \in [0, 3t]$ $\pi_i$ satisfies $m_i \leq m_j + t$. There is no $m_i$ such that $m_i > m_j + t$.

We distinguish two cases.

(a) Suppose $m_j \in (m^*, a]$:

- When $m_i \in [m_j - t, \min\{a, 3t\}]$, we have $\pi_i = \Gamma(m_i)$. $\Gamma(m_i)$ is maximum for $m_i = \Psi(m_j) < m_j < \min\{a, m_j + t\}$, the first inequality being implied by Lemma 3.
- When $m_i \in [a, 3t]$, the profit

$$\pi_i = \Upsilon(m_i) = \max_{\Phi_i, \Omega_i \in [0, 1]} \pi_i(\Phi_i, \Omega_i, m_i) \leq \max_{(\Phi_i, \Omega_i) \in \mathbb{R}^2} \pi_i(\Phi_i, \Omega_i, m_i) = \Gamma(m_i),$$

as the second program is less constrained than the first one. Moreover $\Gamma(m_i) \leq \Gamma(a)$ since $\Gamma(m_i)$ is decreasing for $m_i \geq \Psi(m_j)$ and $a > \Psi(m_j)$.

Thus, $\Upsilon(m_i) \leq \Gamma(a) \leq \max_{[0, a]} \Gamma(m_i)$.

$\pi_i$ reaches its maximum on $[0, 3t]$ either at $m_j - t$ or $\Psi(m_j)$. In both cases, $\varphi_i(m_j) < m_j$.

(b) Suppose $m_j \geq a$. When $m_j - t \leq m_i \leq a$, we have $\Phi_i^* = 1$, $\Omega_i^* = 0$ and $\Phi_j^* = \frac{m_j}{a}$. When $a \leq m_i < 3t(\leq m_j + t)$, we have $\Phi_i^* = \Phi_j^* = 1$ and $\Omega_i^* = 0$.

Thus, the profit of Firm $i$ is given by:

$$\pi_i = \begin{cases} \frac{m_i^2(t-m_j+m_i)}{4at} & \text{if } m_j - t < m_i < a \\ \frac{(t-m_j+t)(m_i-\frac{a}{2})}{2t} & \text{if } a < m_i < \min\{m_j + t, 3t\} \end{cases} \quad (18)$$

- The first expression of Equation (18) is maximal at $m_i = \frac{2(t+m_j)}{3} < m_j$ (since $2t < m^* < m_j$).

\textsuperscript{5}Indeed $\Psi(m_j)$ is the only extremum of the function and it is a maximum since the second order derivative at this point equals $\frac{2at^2(m_j-a)+m_j^2(m_j-2t-2)}{2at^4} < 0$.

\textsuperscript{6}If $a > 3t$ there is no need to consider this interval.

\textsuperscript{7}$m_i = \frac{2(t+m_j)}{3}$ is the only extremum of the function and it is a maximum since the second order derivative at this point equals $-\frac{t+m_j}{2at} < 0$. 

16
\* The second expression of Equation (18) is maximal\(^8\) at \( m_i = \frac{(t + m_j) + a}{2} < m_j \) (since \( t + \frac{a}{2} < a \leq m_j \)).

\[ \pi_i \] reaches its maximum on \([0, 3t]\) either at \( m_j = t, \frac{2(t + m_j)}{3} \) or \( \frac{(t + m_j)}{2} + \frac{a}{4} \). In all cases \( \varphi_i(m_j) < m_j \).

(iii) For \( m_j = m^* \), the profit of Firm \( i \) is increasing up to \( m^* - t \). For \( m^* - t < m_i < \min\{a, m^* + t\} \), \( \pi_i = \Gamma(m_i) \). Considered for unconstrained \( m_i \), \( \Gamma(m_i) \) reaches its maximum at \( \Psi(m^*) = m^* \). Note that \( m^* - t < \Psi(m^*) = m^* < \min\{a, m^* + t\} \). When \( a < m_i < 3t \), the reasoning made in (ii) is valid.

As \( m^* + t > 3t \), for all \( m_i \leq 3t \) we have \( m_i < m^* + t \). The global maximum of \( \pi_i \) on \([0, 3t]\) is thus reached at \( \Psi(m^*) = m^* \).

(iv) For \( m_j \geq m^* \) we have to check that the possible maxima of Firm \( i \)'s profit are increasing in \( m_j \). From (ii) the possible maxima are: \( m_j - t, \frac{2(t + m_j)}{3}, \frac{(t + m_j)}{2} + \frac{a}{4} \) or \( \Psi(m_j) \). The three first expressions are increasing w.r.t \( m_j \). Note that from the proof of (ii) and (iii) \( \Psi(m_j) \) may be a maximum only when \( m_j < a \). It remains to show that \( \Psi(m_j) \) is increasing w.r.t. \( m_j \) for \( m_j \in [m^*, a] \). We have

\[ \Psi'(m_j) = \frac{2\left(-4ta^3 + 4a^2m_j^2 + 4ta^2m_j - 4am_j^3 + m_j^4\right)}{3m_j^2(m_j - 2a)^2}, \]

which has the same sign as:

\[ (-4ta^3 + 4a^2m_j^2 + 4ta^2m_j - 4am_j^3 + m_j^4). \]

Consider the expression:

\[ (-4ta^3 + 4a^2m_j^2 + 4ta^2m_j - 4am_j^3 + m_j^4) + aP(m_j) = m_j^2 \left(m_j^2 - 3am_j + 2a^2 + 2ta\right). \]

This expression is strictly positive\(^9\) as \( m_j \leq a \). Since \( P(m_j) < 0 \) for \( m_j \in (m^*, a] \), then necessarily \( (-4ta^3 + 4a^2m_j^2 + 4ta^2m_j - 4am_j^3 + m_j^4) > 0 \). Hence, \( \Psi'(m_j) > 0 \).

\[ \blacksquare \]

**Lemma 7** if \( a > 2t \), at the unique SPNE, each firm chooses \( m_i = m_j = m^* \).

**Proof.**

1. \((m^*, m^*)\) is the only symmetric equilibrium.

   From Lemma 6 (iii) this choice corresponds to a symmetric SPNE. Lemma 6 (i) and (ii) imply that for all \( m_j \neq m^* \) we have \( \varphi_i(m_j) \neq m_j \) thus no other symmetric equilibrium exists.

---

\(^8\)This function is concave since the second derivative of the function: \( \frac{\partial^2}{\partial m_j^2} \left(\frac{(t - m_j)(m_j - \frac{a}{2})}{2t}\right) = -\frac{1}{t} < 0 \)

\(^9\)The polynomial \( m_j^2 - 3am_j + 2a^2 \) has two roots, \( a \) and \( 2a \), and is positive for \( m_j < a \).
2. There is no asymmetric equilibrium.

Suppose there exists an asymmetric equilibrium such that \( m_j^* = \varphi_j(m_i^*) > m_i^* = \varphi_i(m_j^*) \).

- If \((m_i^*, m_j^*) \in [0, m^*)^2\), Lemma 6 (i) implies \( \varphi_i(m_j^*) = m_i^* > m_j^* \).
- If \((m_i^*, m_j^*) \in (m^*, \infty) \times (m^*, \infty)\), Lemma 6 (ii) implies \( \varphi_j(m_i^*) = m_j^* < m_i^* \).
- If \( m_i^* \in [0, m^*) \) and \( m_j^* \in (m^*, \infty) \), from Lemma 6 (iv), \( \varphi_i(m_j^*) \) is increasing in \( m_j \) whenever \( m_j \geq m^* \). Thus \( m_i^* = \varphi_i(m_j^*) > \varphi_i(m^*) = m^* \).

In all cases, a contradiction results.

\[ \square \]

Case 2: \( a \leq 2t \)

**Lemma 8** Assume that \( a \leq 2t \) and let \( \varphi_i(m_j) \) be the best reply of Firm \( i \) to \( m_j \). We have the following results:

(i) if \( m_j \in [0, t + \frac{a}{2}) \), \( \varphi_i(m_j) > m_j \).

(ii) if \( m_j > t + \frac{a}{2} \), \( \varphi_i(m_j) < m_j \).

(iii) \( \varphi_i(t + \frac{a}{2}) = t + \frac{a}{2} \).

(iv) \( \varphi_i(m_j) > 0 \), \( \forall m_j \in [a, \infty) \).

**Proof.** When \( a \leq 2t \), we have \( a < t + \frac{a}{2} \).

Note that in all cases (similarly to the proof of Lemma 6) when \( m_i \leq m_j - t \), we have \( \Phi_i^* = \frac{m_i}{a} \) as \( m_i < a \). The profit of Firm \( i \), \( \pi_i = \frac{m_i^2}{2a} \) is strictly increasing w.r.t \( m_i \).

(i) Two cases have to be distinguished as the expressions of \( \Phi_i^*, \Phi_j^*, \Omega_i^* \) and \( \Omega_j^* \) are different for \( m_j < a \) and \( m_j > a \).

(a) Suppose \( m_j < a \). It follows (similarly to the proof of Lemma 6 (i)) that the best reply to \( m_j \) is necessarily strictly larger than \( m_i \). Indeed, as \( P(m_j) > 0 \) for all \( m_j \in [0, a] \) when \( a \leq 2t \) (Lemma 3), then \( \Psi(m_j) > m_j \).

(b) Suppose \( a < m_j < t + \frac{a}{2} \). The profit of Firm \( i \) is given by:

\[
\pi_i = \begin{cases} 
\frac{m_i^2(t-m_i+m_j)}{4at} & \text{if } m_j - t < m_i < a \\
\frac{(t-m_i+m_j)(m_i-a)}{2t} & \text{if } a < m_i < m_j + t \\
0 & \text{if } m_i > m_j + t
\end{cases}
\]  

(19)

The first expression of Equation (19) is maximal at \( m_i = \frac{2(t+m_j)}{3} > a \), then \( \pi_i \) is increasing on the first interval. The second expression of Equation (19) is maximal at \( m_i = \frac{t+m_j}{2} + \frac{a}{4} \in [a, m_j + t] \). Thus, \( \pi_i \) reaches a global maximum at:

\[
\varphi_i(m_j) = \frac{(t+m_j)}{2} + \frac{a}{4}.
\]

Moreover \( \varphi_i(m_j) = \frac{(t+m_j)}{2} + \frac{a}{4} > m_j \) when \( m_j < t + \frac{a}{2} \).
(ii) For \( m_j > t + \frac{a}{2} \), the reasoning made for (i)(b) holds and
\[
\varphi_i(m_j) = \frac{(t+m_j)}{2} + \frac{a}{4} < m_j \quad \text{when} \quad m_j > t + \frac{a}{2}.
\]

(iii) Obviously,
\[
\varphi_i(m_j = t + \frac{a}{2}) = \frac{(t+m_j)}{2} + \frac{a}{4} = t + \frac{a}{2}.
\]

(iv) \( \forall m_j \in [a, \infty) \), we have:
\[
\varphi_i'(m_j) = \frac{1}{2} > 0.
\]

\textbf{Lemma 9} if \( a \leq 2t \), at the unique SPNE, each firm chooses \( m_i = m_j = t + \frac{a}{2} \).

\textbf{Proof.} From Lemma 8, \( (t + \frac{a}{2}, t + \frac{a}{2}) \) corresponds to a symmetric SPNE and there is no other symmetric equilibrium. Suppose now that there exists an asymmetric equilibrium \( (m_i^*, m_j^*) \) such that \( m_j^* = \varphi_j(m_i^*) > m_i^* = \varphi_i(m_j^*) \).

Considering all the possible cases: \( (m_i^*, m_j^*) \in [0, t + \frac{a}{2})^2 \); \( (m_i^*, m_j^*) \in (t + \frac{a}{2}, \infty) \times (t + \frac{a}{2}, \infty) \) and \( (m_i^*, m_j^*) \in [0, t + \frac{a}{2}) \times (t + \frac{a}{2}, \infty) \), using Lemma 8 and reasoning as in the proof of Lemma 7, a contradiction results.

The proof of Proposition 1 ends here. \( \blacksquare \)

\textbf{Proof of Lemma 4.}

It is easy to check that: \( \frac{\partial \pi^*}{\partial m} = \frac{m(2(a-m)^2 + am)}{2a^3} > 0 \)

To find the variation of the profit margin at equilibrium \( m^* \) w.r.t the advertising cost \( a : \frac{dm^*}{da} \), we use the implicit functions theorem:
\[
P'_a + P'_m \frac{dm^*}{da} = 0,
\]
which implies:
\[
\frac{dm^*}{da} = -\frac{P'_a}{P'_m}
\]

We have, \( P'_m < 0 \) and
\[
\frac{\partial P(m)}{\partial a} = 2 \left( 4at - 2mt - m^2 \right).
\]

We have
\[
\frac{m \frac{\partial P(m)}{\partial a}}{2} + P(m) = 2a \left( 2at - m^2 \right).
\]
Replacing in this expression \( m \) by \( m = \sqrt{2at} \), the right hand side is null which implies \( P(\sqrt{2at}) = -\frac{m \frac{\partial P(\sqrt{2at})}{\partial a}}{2} < 0 \). Consequently, \( \sqrt{2at} > m^* \), which implies \( 2a \left( 2at - m^2 \right) > 0 \). Hence using Equation (21) for \( m = m^* \), \( \frac{\partial P(m=m^*)}{\partial a} < 0 \) thus \( \frac{dm^*}{da} > 0 \). \( \blacksquare \)
Lemma 10  The sequential equilibrium in which the two firms choose first their prices then their advertising levels coincides with the equilibrium of the game where the firms choose simultaneously their prices and advertising strategies.

Proof. In a simultaneous game, the couple of strategies $((m_i^*, \Phi_i^*, \Omega_i^*), (m_j^*, \Phi_j^*, \Omega_j^*))$ is a Nash equilibrium if

$$ (m_i^*, \Phi_i^*, \Omega_i^*) = \arg\max_{(m, \Phi, \Omega)} \pi_i(m_i, \Phi_i, \Omega_i, m_j^*, \Phi_j^*, \Omega_j^*) . $$

$i = 1, 2; j = 1, 2; i \neq j$

For a given $m_i$, at equilibrium, $(\Phi_i, \Omega_i)$ is necessarily given by:

$$ \Phi_i = \Phi_i^* (m_i, m_j^*) , $$

$$ \Omega_i = \Omega_i^* (m_i, m_j^*) , $$

$\Phi_i^*$ and $\Omega_i^*$ being defined in Lemma 2.

Thus

$$ m_i^* = \arg\max_{m_i} \pi_i (m_i, \Phi_i^* (m_i, m_j^*), \Omega_i^* (m_i, m_j^*), m_j^*, \Phi_j^*, \Omega_j^*) $$

and

$$ \Phi_i^* = \Phi_i^* (m_i^*, m_j^*) , $$

$$ \Omega_i^* = \Omega_i^* (m_i^*, m_j^*) . $$

Note that $\Phi_i^* (m_i, m_j) = \Phi_i^* (m_i)$ for $0 \leq m_i \leq m_j + t$. When $m_i \geq m_j + t$, $\pi_i$ is independent of $\Phi_i$. Note also that $\pi_i$ does not depend on $\Omega_i$. As a result,

$$ m_i^* = \arg\max_{m_i} \pi_i (m_i, \Phi_i^* (m_i), \Omega_i^* (m_i, m_j^*), m_j^*, \Phi_j^*) . $$

We also necessarily have that $\Phi_j^* (m_i, m_j) = \Phi_j^* (m_j)$, thus

$$ m_i^* = \arg\max_{m_i} \pi_i (m_i, \Phi_i^* (m_i), \Omega_i^* (m_i, m_j^*), m_j^*, \Phi_j^* (m_j^*)) . $$

In a sequential game, we solve the choice of advertising strategies $(\Phi_i, \Omega_i)$ first, then the choice of markup prices $m_i$. To do that, we maximize $\pi_i$ w.r.t $(\Phi_i, \Omega_i)$ for given $(m_i, m_j)$:

$$ (\Phi_i^* (m_i, m_j), \Omega_i^* (m_i, m_j)) = \arg\max_{(\Phi_i, \Omega_i)} \pi_i (m_i, \Phi_i, \Omega_i, m_j, \Phi_j, \Omega_j) . $$

To solve the price step, we replace the advertising strategies with their values at equilibrium to obtain firms’ profits which depend now only on $m_i$ and $m_j$.

$$ m_i^* = \arg\max_{m_i} \pi_i (m_i, \Phi_i^* (m_i, m_j^*), \Omega_i^* (m_i, m_j^*), m_j^*, \Phi_j^* (m_i, m_j^*), \Omega_j^* (m_i, m_j^*)) . $$

Moreover recall that Firm i’s profit does not depend on $\Omega_j$.

Thus

$$ m_i^* = \arg\max_{m_i} \pi_i (m_i, \Phi_i^* (m_i), \Omega_i^* (m_i, m_j^*), m_j^*, \Phi_j^* (m_j^*)) . $$
References


